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ABSTRACT

THIS IS ONE OF A SERIES OF UNITS INTENDED FOR BOTH PRESERVICE AND INSERVICE ELEMENTARY SCHOOL TEACHERS TO SATISFY A NEED FOR MATERIALS ON "NEW MATHEMATICS" PROGRAMS WHICH (1) ARE READABLE ON A SELF BASIS OR WITH MINIMAL INSTRUCTION, (2) SHOW THE PEDAGOGICAL OBJECTIVES AND USES OF SUCH MATHEMATICAL STRUCTURAL IDEAS AS THE FIELD AXIOMS, SETS, AND LOGIC, AND (3) RELATE MATHEMATICS TO THE "REAL WORLD," ITS APPLICATICNS, AND OTHER AREAS OF THE CURRICULUM. IN THIS UNIT, THE GROUPING MODEL FOR APPLYING MULTIPLICATION AND DIVISION IS EXPLORED AND EXTENDED THROUGH VARIOUS SUB-MODELS AS AN ILLUSTRATION OF THE CONCEPTUAL MODEL APPROACH TO THE APPLICATIONS OF MATHEMATICS. THE FOUR PARTS OF THIS UNIT ARE (1) MULTIPLICATION, (2) WHOLE NUMBER DIVISION WITHOUT A REMAINDER, (3) THE QUANTITY MODELS FOR FRACTICNS AND MULTIPLICATION, AND (4) A QUANTITY MODEL FOR DIVISION OF FRACTICNS. (RP)

CONCEPTUAL MODELS IN TEACHING THE USES OF NUMBER AND OPERATION

USOE Project

R. G. Clason

INTRODUCTION

Numbers and words are alike in that they are used to describe the physical world. Further, the more enlightened the user, the greater the detail and quality of the information he can be given through each of them. A weatherman can say that it will be partly cloudy tomorrow with a high of 88°. But to a more knowledgeable audience, he may also say that the clouds will be cumulus building to 14,000 ft. with cloud cover increasing from .5 to .8, humidity 78 % and precipitation probability 20%. To obtain the information represented by the numbers used in this more detailed statement required ability to interpret decimal fractions, percentages, and some idea of what is meant by probability.

In the most fundamental number situations, the relation of the number to the physical world is direct. The number 4 can be directly associated with a physical set of apples. But as the study of number proceeds to more involved situations, this relation is sometimes pushed into the background. In learning the basic addition facts, it would be inefficient for a child to phrase each occurence of each combination in terms of pennies or apples. Learning to add fractions is involved enough so that insisting on a physical interpretation for each fraction at each step would be confusing. Further, emphasis on the inner structure of mathematics often suggests explanations based on mathematical laws rather than explanations based on physical situations. For example, the fact that $\frac{3}{4} = \frac{6}{8}$ can be demonstrated mathematically without recourse to any concrete situation at all by appealing to abstract laws and definitions:

$$\begin{aligned} \frac{3}{4} &= \frac{3}{4} \times 1 && \text{identity law} \\ &= \frac{3}{4} \times \left(2 \times \frac{1}{2}\right) && \text{inverse law} \\ &= \frac{3}{4} \times \frac{2}{2} && \text{definition of multiplication} \\ &= \frac{6}{8} && \text{definition of multiplication} \end{aligned}$$

Such de-emphasis on the real world allows more mathematically precise treatment of numbers, but it also requires added concern that the use of numbers be made clear. There is limited value in teaching a student to divide two fractions if he remains incapable of recognizing in what situations such a division can be used.

MODELS

In learning to use numbers, a child developes certain mental patterns which allow him to see similarities among concrete situations and apply general information already discovered to particular cases. These patterns have been called variously schema, models, and constructs. We will use the word "model" in discussing these patterns which are developed in learning, since this word is suggestive of the uses made of them in arithmetic, in spite of the fact that "model" is in current use in several different contexts. A person may use a model which he holds as a remembered visual image in order to identify a figure as a square. Students develop a model for an "unknown" which allows them to think about addition in situations like $4 + ? = 7$, even though they don't know the second addend.

A model is used in various ways. It allows identification. John has 11 ¢. He spends 4¢. How much has he left? A student who has a working "take away" model for subtraction will identify this as a subtraction problem;[1] 4¢ is being taken away from 11 ¢, therefore subtraction is the correct procedure. A model can be used to justify an algorithm. In the subtraction algorithm, the need for borrowing is justified using the "take away" model. For example, in

$$\begin{array}{r} 725 \\ -682 \\ \hline \end{array}$$

we must borrow in the ten's column because 8 tens can't be taken away from 2 tens. A simple model may be used in explaining part of a more complicated situation. Subtraction is involved in the division algorithm.

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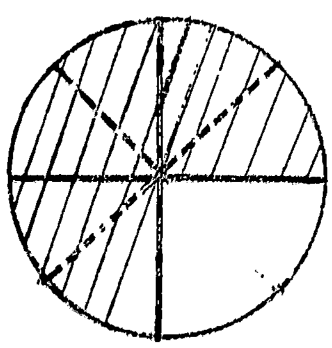
1. For briefness, the word "problem" will be used only to refer to real world problems. Thus we won't call $11 - 4 = ?$ a problem. This is only a convenience, and does not represent common usage.

In the division

$$\begin{array}{r} 5 \\ 3\overline{)15} \\ \underline{15} \\ 0 \end{array}$$

we subtract to see that when five threes are <u>taken away</u> from 15, it leaves nothing. Models can be used to back up abstractly stated laws of arithmetic and to verify and reinforce results obtained through abstract reasoning. For example, the abstract argument used to prove that $\frac{3}{4} = \frac{6}{8}$ can be verified by looking at 3/4 of a circle, and then subdividing the quarters into eights. Now, rather than appealing to precise



mathematical laws to show that $\frac{3}{4} = \frac{6}{8}$, we appeal to the imprecise but perceptual maxim: no matter how you cut it, it's the same pie.

A child chooses from a broad range of patterns as he builds his own collection of models. He may use a remembered picture or an explanation given to him by a teacher for some particular problem. A teacher's decision to suggest a particular approach to a problem either to a group or in giving individual help can thus be seen as a crucial one.

As we consider learning with attention to the formation of models, we are lead to inquire about the advantages and disadvantages of various types of models. It is useful for a model to be general, that it apply to a large class of problems. But a proposed model can be so general that a child will fail to see its application. To tell a third grader "To solve this problem, multiply 4 by 3," helps only with one problem. But at the other extreme, to tell him, "In solving problems, always take into account all possible cases," probably won't help him at all. Even the most basic models have some built-in difficulties. "Take away" will help solve a lot of problems and aid in explaining the subtraction algorithm, but students need help in seeing why problems which ask "How many more?" are instances of "take away", and it is hard to justify $9 - (-2) = 11$ with this model. A student who accepts the attitude that all subtraction is "take away" will meet difficulties in rationalizing a number of other situations in which subtraction is needed.

But in spite of occasional spots that require special attention, there are a limited number of basic models which can be applied generally enough to use them as a hinge on which to swing most arithmetic problems. Because of the power of these basic models, it seems worthwhile to examine them carefully. In the following units the grouping model, a very basic model for applying multiplication and division, is explored and extended through various sub-models" as an illustration of the conceptual model approach to the application of mathematics.

CONCEPTUAL MODELS

IN

TEACHING THE USE OF NUMBER AND OPERATION

USOE Project

R. G. Clason

Unit 1. Multiplication

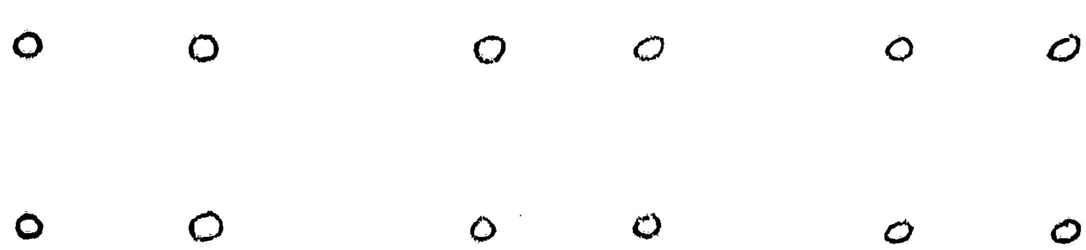
THE GROUPING MODEL

The fundamental model for multiplication is the grouping model: to find the total number of objects in a set of equal groups, multiply the number of groups by the number in each group. Henry has 3 bags of marbles with 4 marbles in each bag. How many marbles has he ? There are three numbers in this problem, each with a distinct function:

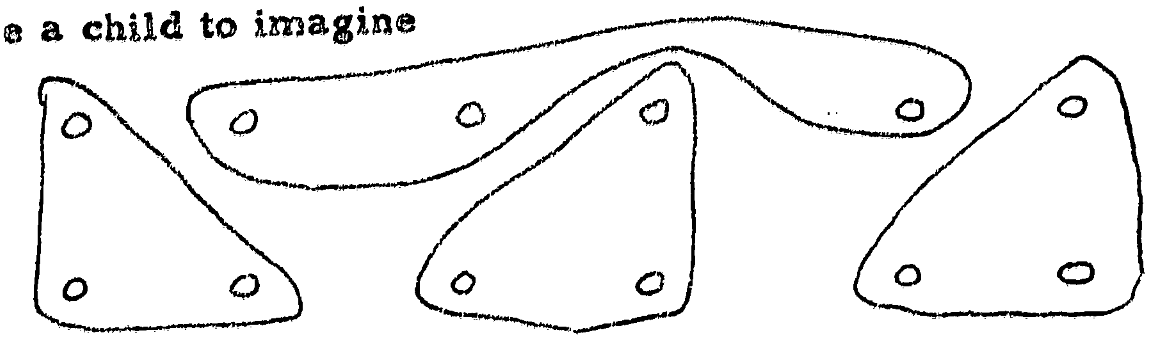
3	x	4	=	?
number of groups		number in each group		total

Whenever a grouping problem in resolved into a multiplication sentence, we will maintain this order, the first factor representing the number of groups, the second the number in each group. This is in keeping with the old wording, 3 fours are 12.

A teacher who maintains a convention such as this, at least in explanations, can avoid some confusion to her students. Finding four three's that is, solving $4 \times 3 = ?$ will produce the correct answer to the problem above, but saying "four threes" over the picture

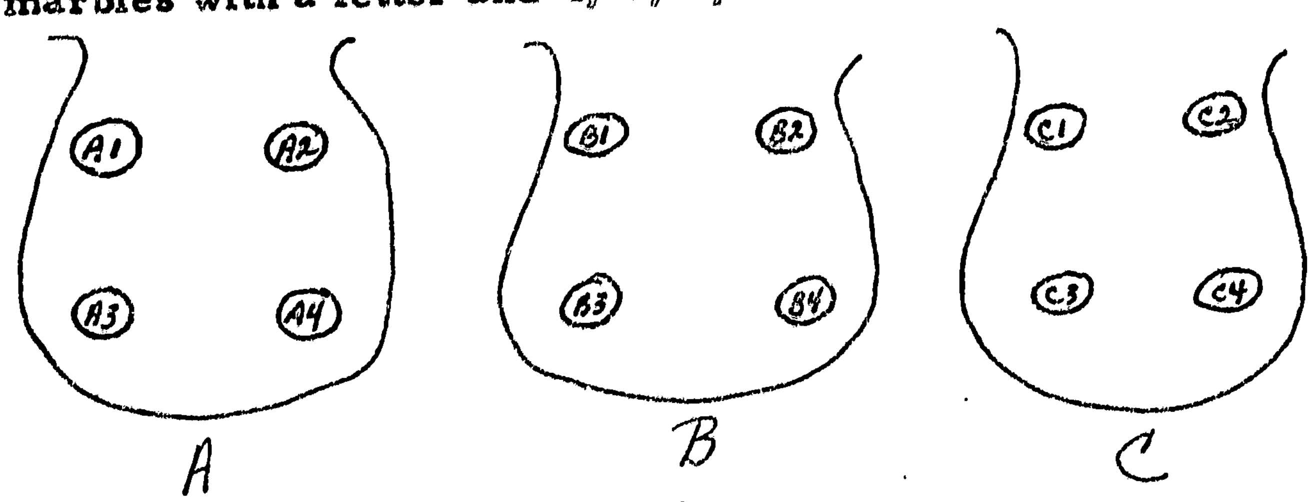


may force a child to imagine

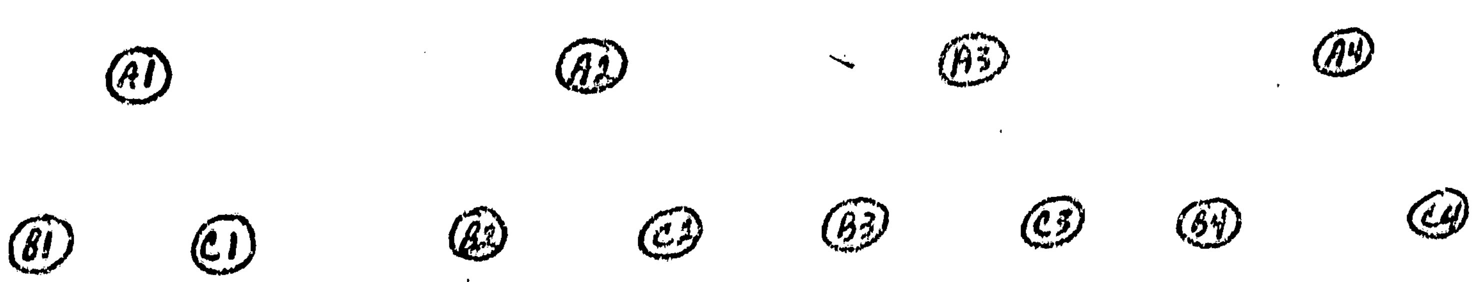


which is correct, but may cause unexpected difficulty when there are seven marbles in each of three bags.

As the unorthodox grouping of marbles pictured above suggests, there is no single proper way to solve a grouping problem. In fact, any problem that can be resloved into $a \times b = ?$ can also be resolved into $b \times a = ?$. For illustration, consider the marble problem above, which seems to be natural for $3 \times 4 = ?$. Label the bags A, B, and C, and the marbles with a letter and 1, 2, 3, and 4 as follows:

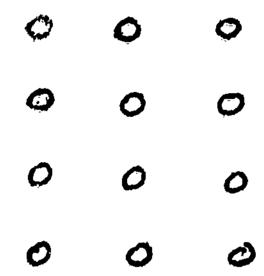


Now we can mentally group by number,

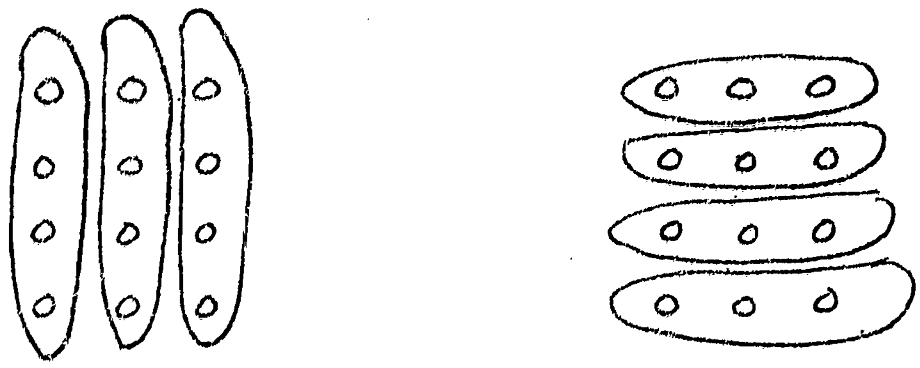


which is a $4 \times 3 = ?$ grouping.

There is also, of course, the diagramatic model



which can be viewed two ways, either as 3 x 4 or as 4 x 3 and thereby



serves as a model for the commutive property which states that for all whole numbers a and b , a x b = b x a .

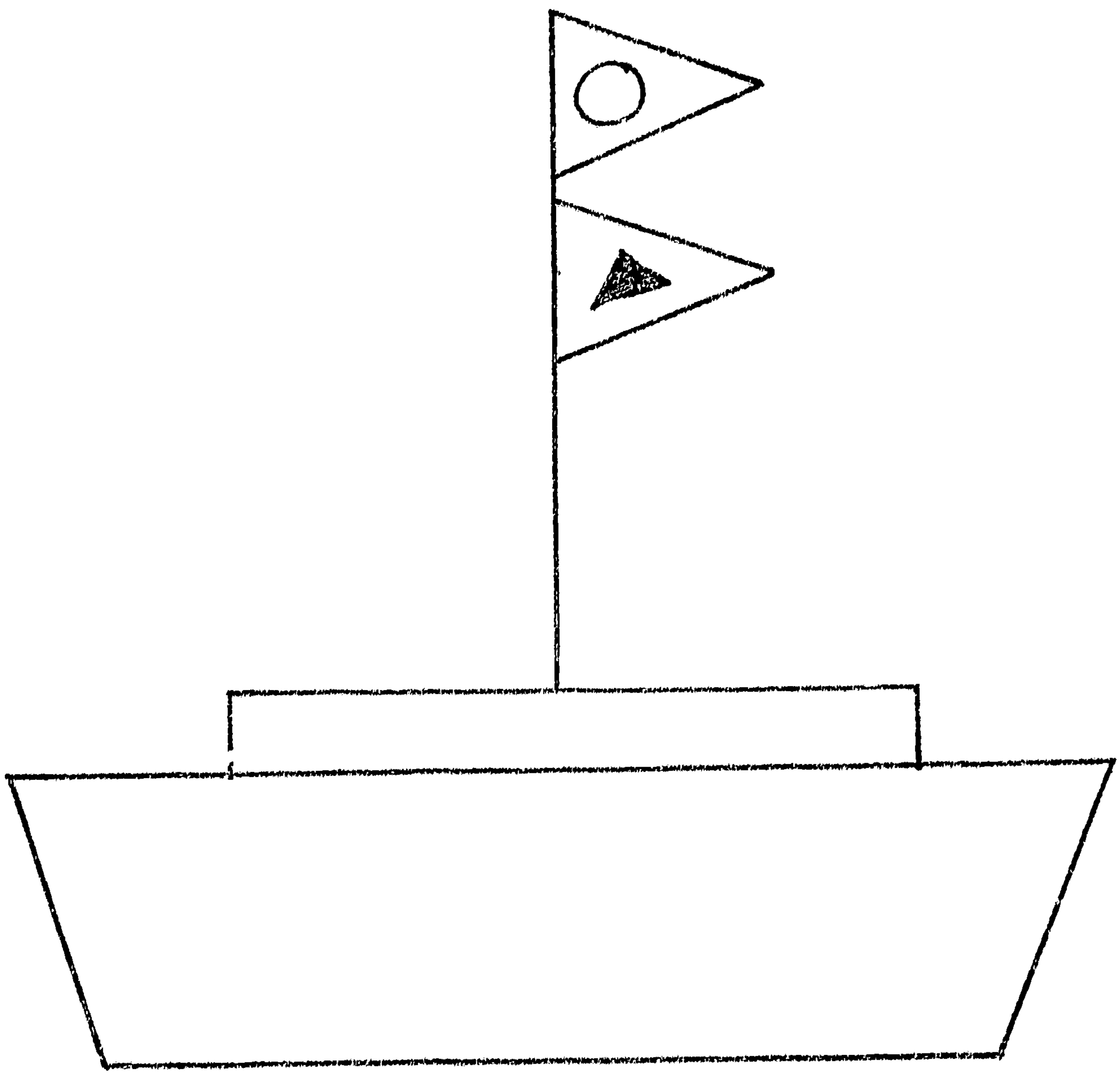
EXERCISES

1. Draw a picture showing a grouping of 3 bags of 7 marbles each which is described by 7 x 3 = ?.

2. Explain why the area of a 3" by 4" rectangle is 12 square inches in two different ways, using two different groupings.

3. A deck of cards is dealt into 4 equal hands with 5 cards in each hand. Explain how this problem can be resolved into both 4 x 5 = ? and 5 x 4 = ?.

4. A boat owner has room to hoist two flags on his mast. If he has four flags, all different, how many different two flag signals could he display?



DIAGRAMS SHOWING GROUPING

There are many ways of showing grouping situations. The device chosen to illustrate a particular problem depends on the nature of the problem, and on the possible power of the illustration in analyzing more involved problems. A few ways of picturing grouping are shown and briefly discussed below.

(1) Proximity grouping

```
   x          x
  x x        x x
```

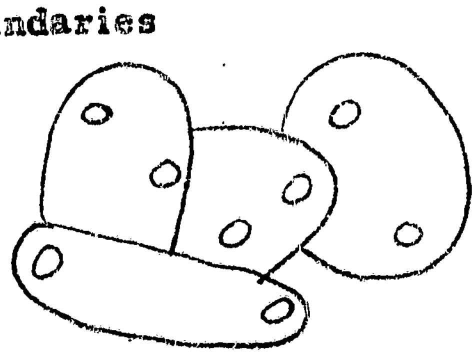
$$2 \times 3 = ?$$

This is probably the simplist way of picturing groupings. It can be extended to grouped groups, to illustrate three factor multiplication.

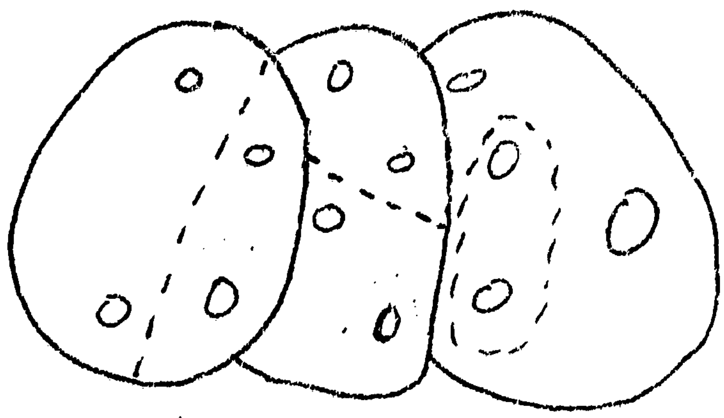
```
         x x                          x x
         x x                          x x
   x x        x x               x x        x x
   x x        x x               x x        x x
```

$$2 \times (3 \times 4) = ?$$

(2) The use of boundaries



Boundaries without proximity grouping can be useful in teaching division as the inverse of multiplication. For example, in constructing the figure above, draw all 8 elements before making the boundaries to show the division $? \times 2 = 8$. (More on this later.) Grouping groups can also be illustrated using different kinds of boundaries.



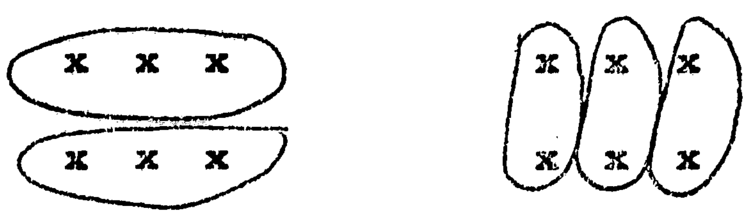
(3) Rectangular arrays

x x x

x x x

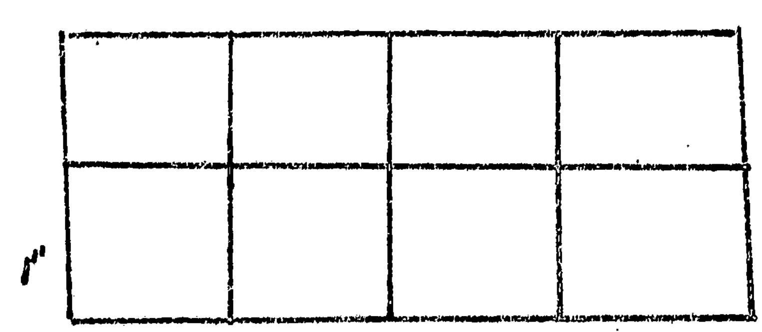
2 x 3 = ?

A rectangular array, grouped first by rows and then by columns, illustrates the commutative law for multiplication.



2 x 3 = 3 x 2

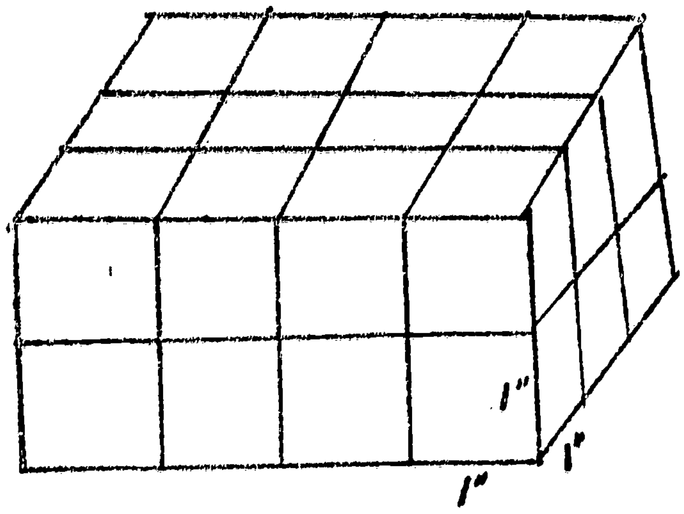
(4) Area



$2 \times 4 = ?$

There are 2 strips with 4 square inches in each strip.

(5) Volume



Finding volume is an example of grouping groups.

a. Count 4 cubic units in the bottom row of the front stack.

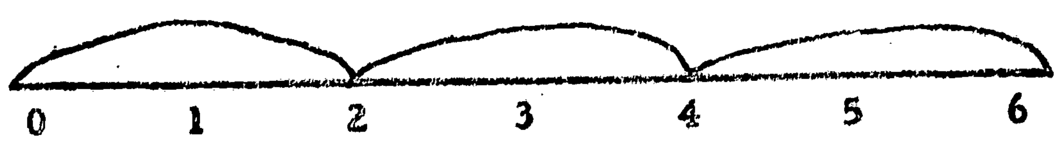
b. There are 2 such rows in the front stack, $2 \times 4 = ?$

c. There are three such stacks in the whole block,

$$3 \times (2 \times 4) = ?$$

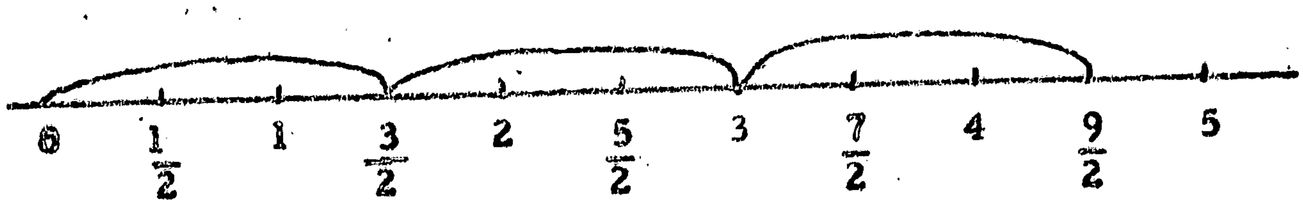
(6) Number line

Quantities can be represented by lengths on the number line, with groups of equal length depicted by humps above the line.



$$3 \times 2 = ?$$

Number line pictures can also be used in fractional situations (to be discussed more later).



$$3 \times 1\frac{1}{2} = ?$$

CROSS PRODUCT

Two related grouping devices, which can be used more easily in certain abstract problem than those listed above, are described in this section and in the one following. Consider a problem: A spy kit contains 4 false moustaches and 3 false beards. How many different disguises can a spy possibly wear, using this kit?

This problem is not quite so directly classed as a grouping model multiplication as the marble problem in the first section. But we can see the groupings if we name the various moustaches and beards,

moustaches	beards
1	a
2	b
3	c
4	

and group the possibilities by moustaches:

1a		2a		3a		4a	
1b	1c	2b	2c	3b	3c	4b	4c

$$4 \times 3 = ?$$

In the language of set theory we have formed, in this solution, the cross product of two sets, $\{1, 2, 3, 4\}$ and $\{a, b, c\}$.

Definition: The <u>cross product</u> of two sets is the set of all ordered pairs that can be formed by choosing the first element of an ordered pair from the first set and the second element of the pair from the second set. For example,

if $S = \{1, 2, 3, 4\}$ and $T = \{a, b, c\}$, then

$S \times T = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, b), (4, c)\}.$

As the grouping argument used to solve the problem indicates, the number of elements in a cross product is the product of the numbers of elements in the two sets. This can also be seen by arranging the elements of the cross product in a rectangular array:

1a	2a	3a	4a
1b	2b	3b	4b
1c	3c	3c	4c

Why is the grouping pattern easier to see in the marble problem than in the disguise problem? One reason is that in the first case, we are grouping concrete objects. Even when the symbols, A2, B1, etc., are attached to the marbles it seems only a device to aid in keeping tract of what marble goes into what group. On the other hand in the disguise problem, we are concerned with possibilities, not concrete objects. The marble problem can be solved by obtaining 3 bags of 4 marbles each and counting marbles. But resorting to an actual spy kit does not allow the simultaneous physical formation of the set of all disguises which one must count to answer the question. Attempts to solve the problem this way can lead to at most 3 disguises, for example 4c, 2b, and 3a, and some spare moustaches. The cross product solution described above is built on an abstract set of ordered pairs, whose elements represent hypothetical possibilities. By forming the abstract cross product set, $S \times T$, we can think simultaneously about the elements (2, a) and (3, a) without being concerned about the physical impossibility of the simultaneous existence of the corresponding disguises.

EXERCISES

1. Solve the marble problem of the first section by forming two sets and then considering their cross product.

2. In deciding what to wear, a girl finds that she can choose from 5 sweaters and 6 skirts. Ignoring color clashes, find how many different combinations are possible. Explain your procedure.

3. A man plans to drive from New York to Denver via Chicago. He discovers 3 acceptable routes from New York to Chicago and 4 from Chicago to Denver. In how many different ways can he make the trip? Explain.

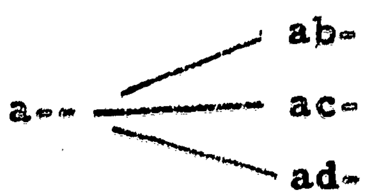
TREES

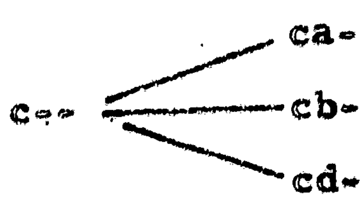
Problems to which an ordered pair (or ordered set of n elements) model apply do not always fall nicely into a cross product pattern. The solution of the following problem illustrates a use of ordered triples which is not a triple cross product: Find the number of different 3 flag signals a sailor can run up his mast if he has 4 different flags.

Name the flags a, b, c, and d. Any one of the flags can be chosen to be the first one run up the mast. We can separate the set of all possibilities into four groups, depending on which flag is on top, a, or b, or c, or d. $4 \times ? = \text{total}$

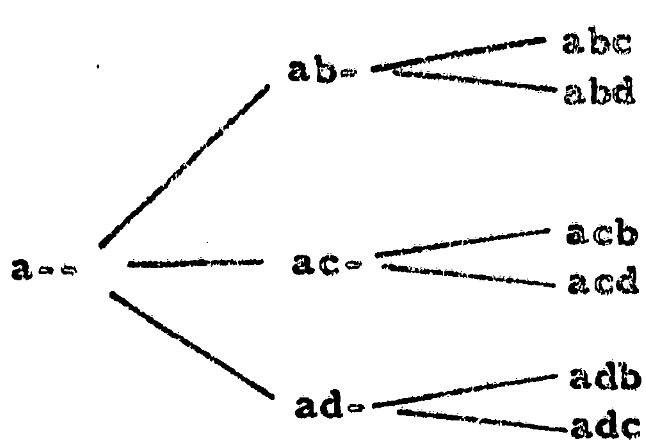
(This is a tentative step; we don't know at this point in the solution how many possibilities there are in each group. It is not even totally obvious that the four groups will have the same number.)

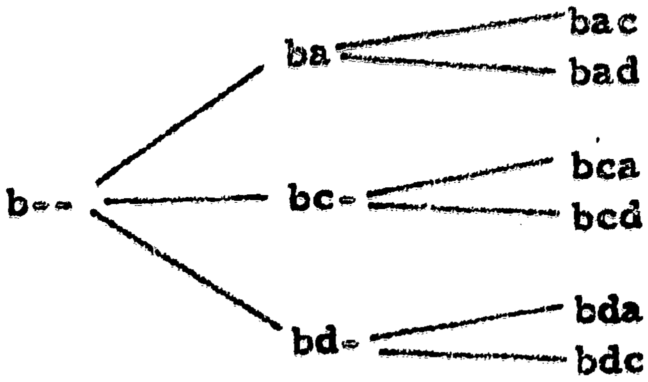
Now each of the four groups can itself be grouped by the flag that appears next to the top.

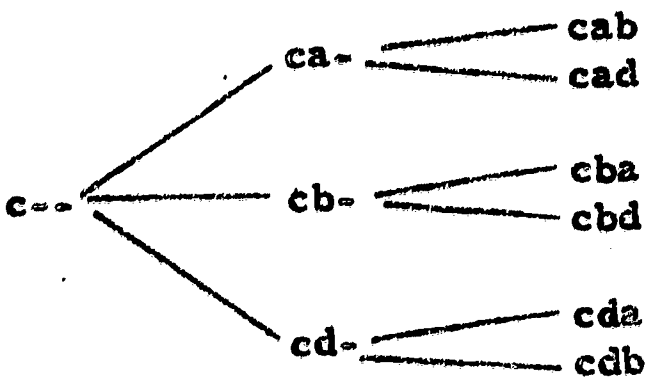


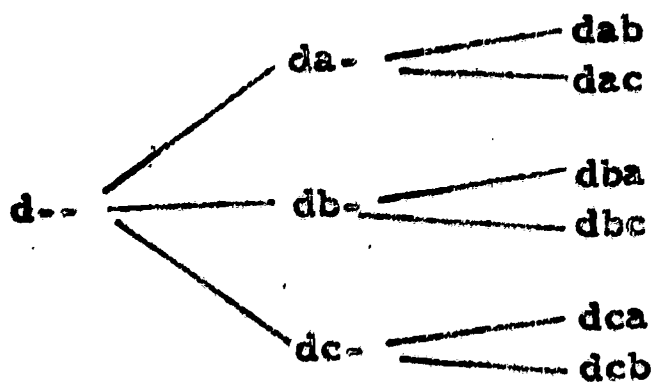


Finally, we can see that each of these groups has exactly two elements.









$4 \cdot \times (3 \times 2) = 4 \times 6 =$ total number of 3-flag signals.

If we let $S = \{a, b, c, d\}$, is this final set of ordered triples the triple cross product is $S \times S \times S$? No, because elements such as (a, b, a), (b, b, c), and (c, c, c) of the cartesian or cross-product, $S \times S \times S$, do not represent possible flag signals.

GROUPING GROUPS

It is interesting to compare the technique used in building trees with that used earlier in finding a volume. In building the tree, we started thinking about the set of all possibilities by noting that the whole set can be broken up into four groups, although the exact nature of the groups was a bit foggy. We hoped to find the other factor in $4 \times ? = \text{total}$. In turn, we found a partial answer to this question in $4 \times (3 \times ?) = \text{total}$. Thus we produced a series of questions, in which each answer depends on a satisfactory answer to the following question until the very last question.

In contrast, in the volume problem, we work at each step with only a portion of the whole set of unit blocks. But at the end of each step, we have a complete answer for some question, for example, at the end of the second step, we know that there were 2×4 units in the front stack.

Generally speaking in grouping situations, it seems easier to proceed as in the volume problem, where each sub-question in fully answered before the next is undertaken. However, without something concrete like the image of a block made from unit cubes, it can be difficult to single out a sub-portion of the set whose number is sought which can be easily generalized as a prototype (for example a stack) for a set of equal groups.

EXERCISES

1. A man has three cans of paint, one read, one blue, one yellow. He intends to paint his son's tricycle one color and his wagon another using this paint. How many different results are possible? Does your solution involve the cross product of two sets?

2. On Michigan passenger car license plates, a pair of letters are used as the first two characters and they are followed by four digits, as for

example in UM 1817. How many different letter pairs can be used? Can a cross product be used in the solution of this problem?

3. How many different 4 flag signals could be hoisted on a mast, if 4 different flags are available?

5. Solve the flag problem of the section on trees, starting by finding the number of possibilities in some definite subset of the set of all possibilities.

6. Solve the volume problem, starting by sub-dividing the whole block of unit cubes into a group of groups.

CONCEPTUAL MODELS

IN

TEACHING THE USES OF NUMBER AND OPERATION

USOE Project

R. G. Clason

Unit 2. Whole number division without a remainder

INTRODUCTION

Division is the inverse of multiplication, that is, a division question is a multiplication statement in which the product and one factor are known, and the other factor is sought. Thus division problems can be reduced to multiplication statements with one factor unknown. Since we have emphasized separate functions for the two factors in multiplication, the first representing the number of groups, the second the number in each group, we are lead to two types of division problems.

MEASUREMENT DIVISION

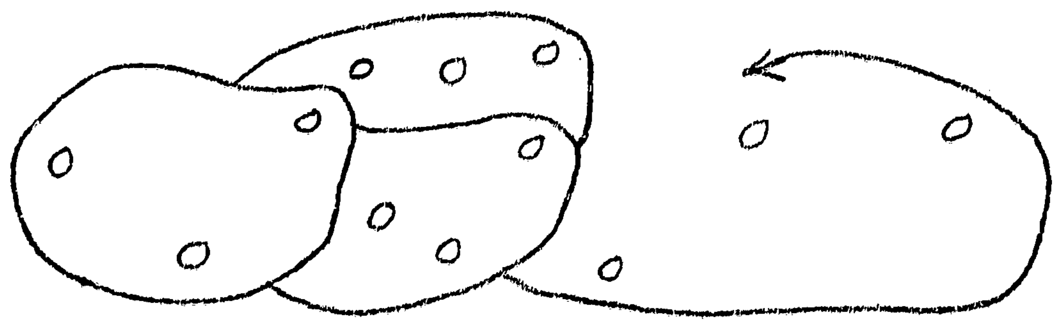
Division problems in which the number in each group is known and the number of groups is sought are called measurement[1] division problems. How many groups of 3 marbles each can be make from 12 marbles?

$$? \times 3 = 12$$

---

1. The word "measurement" is suggested by the "measuring" of the total set in terms of sets whose cardinal number is the second factor.

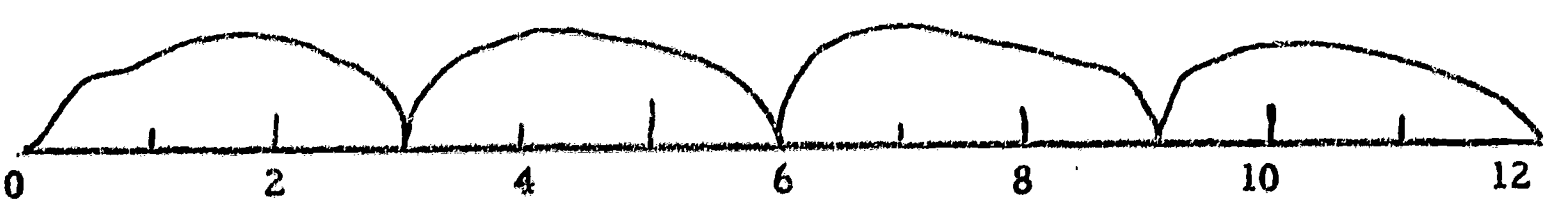
This problem can be solved by direct appeal to boundary grouping, by grouping three marbles at a time until all are encircled and then counting the number of groups.



? X 3 = 12

The same general technique can be used on the number line by counting 3 units in each hump, stopping at 12 units, and then counting humps.

? X 3 = 12



This process of taking out groups of a given size to exhaust a total set is a process of repeated subtraction, and is used in explaining the division algorithm. Removing 3's from 12 one at time,

$$\begin{array}{r} 3\,\overline{)\,12} \\ \underline{-\,3} \\ 9 \\ \underline{-\,3} \\ 6 \\ \underline{-\,3} \\ 3 \\ \underline{-\,3} \\ 0 \end{array}$$

4 3's are 12

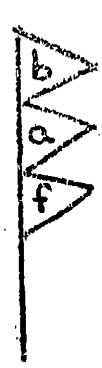
we can count the number of 3's in 12.

When we become more efficient and remove more than one 3-group at a time (we group 3's), we need a special column to keep track of them.

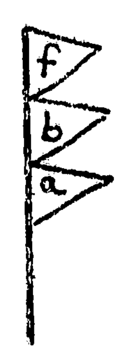
$$? \times 3 = 4269$$

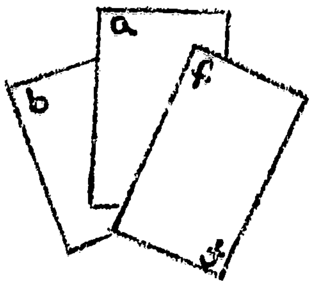
	Number of threes		
3 ⟌ 4269			
-3000	1000		
1269			
- 900	300		
369			
- 300	100		
69			
- 69	23		
0	1423	3's in	4269.

As an example of a problem in which the measurement pattern in less evident, consider the following: How many different 3 card hands are possible from a deck of 7 different cards? This problem is somewhat similar to the flag problems given earlier. If we were looking for possible 3 flag signals from 7 different flags, the answer would be 7 X 6 X 5; 7 possibilities for the top flag, 6 for the next, and 5 for the bottom flag (if this is not clear, think of a tree model). However, we cannot carry this analysis directly over to the card problem. Naming the cards a, b, c, d, e, f, and g, we see that while

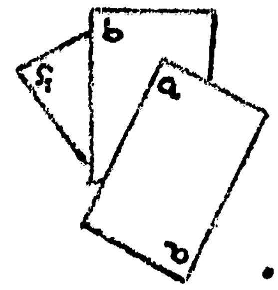


is a different signal from





is the same hand as



Of these two different types of possibilities, the flag possibilities are arrangements since arranging the flags differently gives a different possibility, and the card hands are called combinations since a different combination of cards, without regard to their arrangement, is required to have a different possibility.

Now, the following grouping statement is true:

[The number of arrangements of each combination] X [the number of combinations (the answer to the problem)] = [the total number of arrangements.]

To show a few of the groups involved in this multiplication,

ARRANGEMENTS OF ONE HAND	ARRANGEMENTS OF ANOTHER	ARRANGEMENTS OF A THIRD HAND	
abc	abd	abe	
acb	adb	aeb	
bac	bad	bae	
bca	bda	bea	
cab	dab	eab	
cba	dba	eba	and so forth.

Finding the number in each group, i.e. the number of arrangements per combination, is a separate arrangement problem (like finding how many different 3 flag signals can be made from 3 different flags). In our problem, the answer is 6. Thus, the multiplication statement above becomes the measurement division,

6	x	?	=	7 x (6 x 5)
arrangements per combination		combinations		total arrangements.

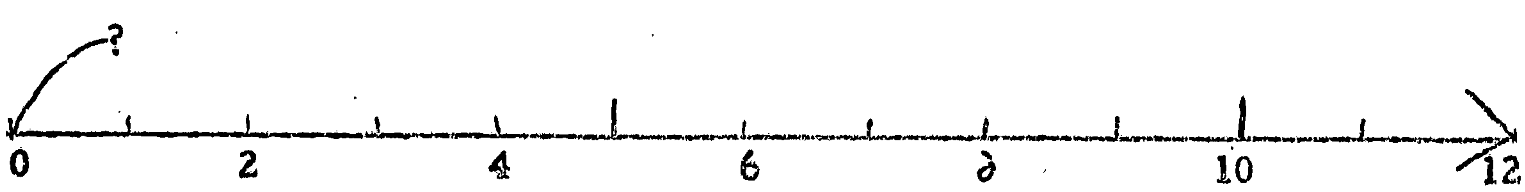
Since $7 \times (6 \times 5) = 7 \times 30 = 210$, the final result, the replacement for "?", is $210 \div 6 = 35$.

PARTITION DIVISION

Division also is needed when we know the number in the product and the number of groups. In this case the number in each group is sought this is called "partition division". How many marbles in each group if I divide (partition) 12 marbles into 3 equal groups?

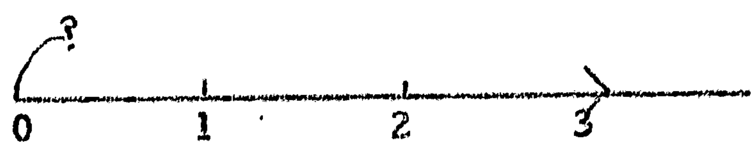
$$3 \times ? = 12$$

Children usually have more difficulty with partition division problems than with measurement problems. One reason for this is that the division algorithm is built up from the measurement model by removing sets of known size from a total one at a time. The solution to a partition division problem cannot be directly perceived in this way. In a picture of 12 marbles, we cannot draw a boundary around one of the groups if we know only that there must be 3 equal groups. To draw the boundries, we need to know how many there are in each group, which, unfortunately, is the answer we are trying to get at. On the number line, the situation is the same;



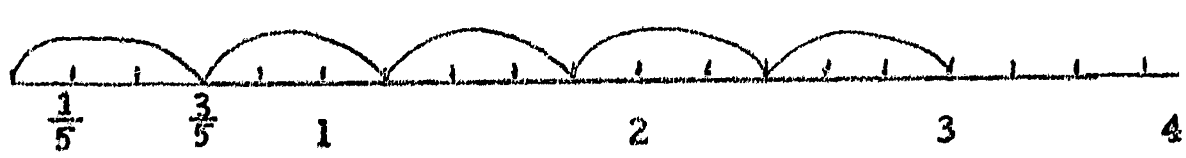
we must draw 3 equal humps without knowing how large to make each one.

Although thinking about pictures illustrating partition division does not lead directly to an answer, a familiarity with the partitioning pattern allows identification of a problem as one requiring division. Further, properties of numbers can be illustrated through the partition model. For example, the fact that $3 \div 5$ and $\frac{3}{5}$ represent the same number can be illustrated through partition division. On the number line, in trying to solve $? \times 5 = 3$, we search for the length of a segment having the property that 5 of them will be 3 units long.



Will a segment of length $\frac{3}{5}$ units work? We try it and see (after first marking off each unit into $\frac{1}{5}$'s).

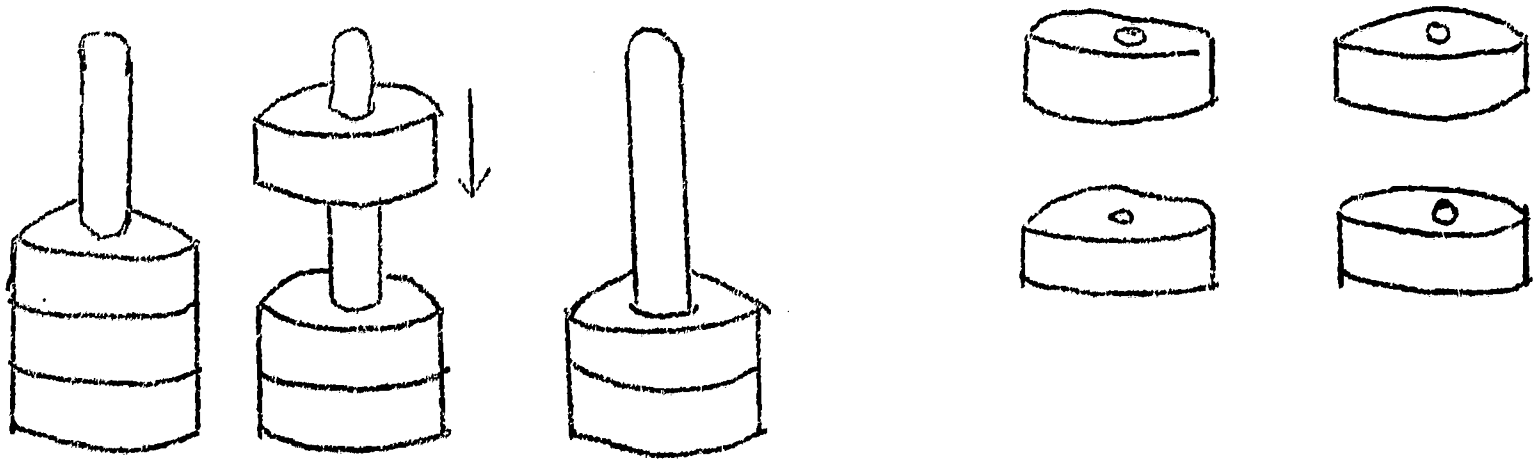
$$5 \times \frac{3}{5} = 3$$



Thus a setment whose length is 3 of 5 equal sub-divisions of a unit (the usual discription of $\frac{3}{5}$) is also the length of a segment 5 of which is 3 units (the answer to $5 \times ? = 3$).

THE INTERRELATION OF THE PARTITION AND MEASUREMENT MODELS

As it was possible to perceive the elements in a rectangular array as grouped either into rows of columns, so it is possible to interprete division problems through either the partition or measurement model. To illustrate this, let us solve what appears to be a partition problem using a measurement analysis. Problem: 12 rings are to be separated into 3 equal groups. How many rings are there in each group? $3 \times ? = 12$ We solve this partition problem by putting the rings on three stakes, one ring at a time on each stake in turn until all the rings have been distributed. A count then shows 4 rings per stake.



On the other hand, the answer to the partition question, "how many on each stake?," could have been found by the measurement device of first making layers (groups) with three rings in each (one such layer is the set of rings at the bottom of stakes). By measuring the set of 12 rings using a "layer" as a unit one could again determine that there would be 4 layers. $? \times 3 = 12$, $? = 4$. The interrelation of the two groupings, layers and stakes-full, is made in recognizing that 4 layers (groups) in the layer (measurement) grouping means 4 on each stake (in each group) in the state (partition) grouping.

As another example of a partition problem solved using the measurement model, consider the following problem and explanation given by a master teacher of arithmetic:

"A boy having 32 apples wished to divide them equally among 8 of his companions; how many must he give them apiece?

If the boy were not accustomed to calculating, he would probably divide them, by giving one to each of the boys, and then another, and so on. But to give them one apiece would take 8 apples, and one apiece again would take 8 more, and so on. The question then is, to see how many times 8 may be taken from 32; or, which is the same thing, to see how many times 8 is contained in 32. It is contained four times. Ans. 4 each." *

In analyzing the ring problem, our ability to perceive the rings in two different but interrelated grouping patterns is facilitated by the ease with which we can physically or mentally move the rings around. We have no pre-conception of any natural physical arrangement of the rings. For contrast, consider the partition problem of a man who wants to cut a 12 foot board to make a bottomless 4 sided sandbox. How long should each side be? One hesitates to use a measurement explanation which involves, even mentally, cutting the whole board into one foot lengths in order to make 4 stacks of 3 boards each, one stack for each side of the sandbox.

* Warren Colburn, Arithmetic upon the Inductive Method of Instruction, Boston: Jordan, Swift, and Wiley, 1845, pp. 142 .

What should be recognized in the ring argument, as in the row and column groupings of a rectangular array, is that what we are doing is verifying the commutative law for multiplication in a particular context. Stated in terms of a missing factor, this law says that, for example, solving $? \times 3 = 12$ also produces the answer to $3 \times ? = 12$, and vice versa. Thus the algorithm for division, developed through measurement division, also solves partition problems. Thus the ultimate goal of achieving and understanding a general and abstract perception of the nature of multiplication and division with numbers, may be approached by using different models, bearing in mind and ultimately pointing out their relations to one anouther as well as to the operations as generalized abstractions.

EXERCISES

1. Make up 6 situations requiring division, 3 which are easily analyzed using a measurement approached and 3 using a partition approach.

2. Give an example of a division problem for which you would use a number line explanation, and one for which you wouldn't.

3. Draw a number line picture for $? \times 5 = 10$ and for $5 \times ? = 10$.

4. Draw a number line picture and use it to explain why $\frac{7}{8}$ and $7 \div 8$ represents the same number. Do the same for $\frac{8}{7}$ and $8 \div 7$.

5. Have you taught children who had trouble with division problems? What do you think caused the difficulty?

6. A deck of 52 cards is dealt into 4 equal hands. How many cards are in each hand? Explain the interrelation of partition and measurement using this problem.

7. How many different poker hands are possible? (5 card hands from a deck of 52 different cards.) How many of these are all hearts? (5 card hands from 13 hearts.) How many different poker hands have all 5 cards from the same suit?

8. The solution to the card problem at the end of the section on measurement division may seem a bit awkward, since it requires the computation of a large number of arrangements in order to find a smaller number of combinations. Try a direct multiplication grouping attack on this problem using a tree. What difficulties do you encounter?

9. Explain why the division of apples problem given by Warren Colburn is a partition problem, and why his solution of it is a measurement solution. Using this problem, explain the interrelation of the two models, i.e., explain how finding the number of groups in one situation gives the number in each group in another related situation.

CONCEPTUAL MODELS IN TEACHING THE USES
OF NUMBER AND OPERATION

USOE Project

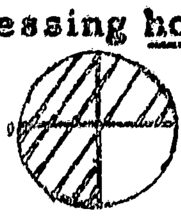
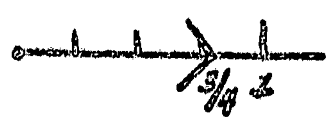
R. G. Clason

Unit 3. The Quantity model for Fractions; Multiplication

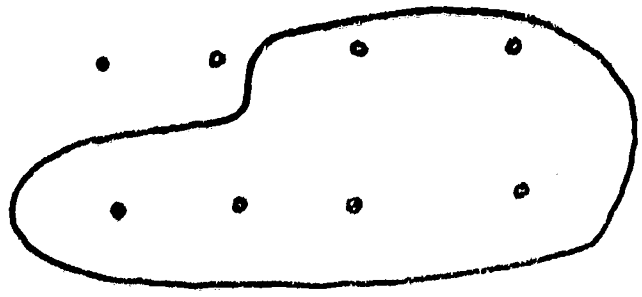
INTRODUCTION

There are various models for relating fractions to the physical world. We could agree that $\frac{3}{4} = ?$ means $4 \times ? = 3$, and explain fractions through partition division. Or we could agree that $\frac{3}{4}$ describes a relation between two sets, the relation that for each 3 elements in one set there are 4 in the other. However, the most used model for fractions is the quantity model.

THE QUANTITY MODEL FOR FRACTIONS

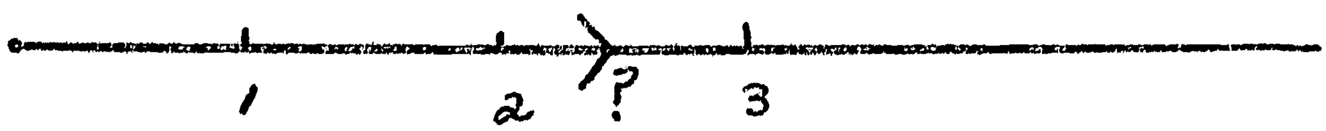
The quantity model for fractions extends the idea of expressing how many with a whole number to expressing how much with a fraction. $\frac{3}{4}$ can express how much pie, , or how much length 

In expressing how much, a fraction involves 3 separate quantities and associated numbers. First a predetermined unit, with respect to which the quantity is to be measured, corresponds to the number 1. This seems almost automatic, particularly in talking about fractions of apples or pies. But as situations become more involved, failure to keep track of the unit with respect to which a quantity is described can cause confusion. For example, when we begin a problem by using $\frac{3}{4}$ of the dots in the accompanying drawing the "unit" is the set of all the dots and $\frac{3}{4}$ describes the circled dots. However, when in the course

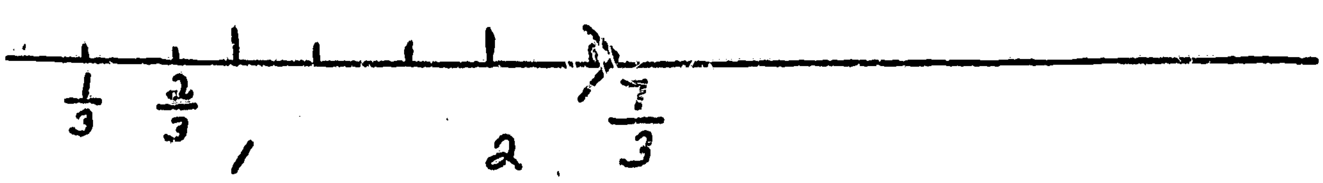


of the solution we use the fact that there are eight dots in the diagram and six of them are circled we are using two different units at once, the entire set and the individual dots. Clearly $\frac{3}{4}$ does not equal 6, although both numbers describe the circled dots. If a student does not understand the root of this seeming paradox, he may have difficulty completing his solution.

The second quantity involved in a fraction, expressed through the denominator, is the key to the extension of the number system from whole numbers to fractions viewed as representing quantities. We cannot be very precise in measuring the length of a line



using only standard units, so a secondary unit or <u>counter</u> is introduced. We cut the unit into, for example, 3 equal pieces and agree to use the length of each as a counter and to describe its length with the new number, $\frac{1}{3}$



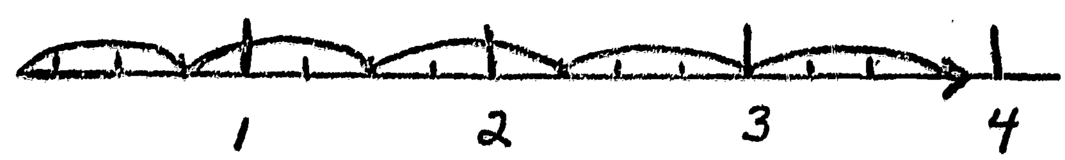
The introduction of such counters is equivalent to assuming numbers which are the answers to partition divisions of 1. The fundamental property of $\frac{1}{3}$ is that $3 \times \frac{1}{3} = 1$. Abstractly, this is expressed by saying that we have created a multiplicative inverse for each counting number, i.e. a number by which we can multiply and get 1. From a quantity model standpoint $\frac{1}{3}$ represents one of the 3 equal parts into which a unit has been divided. The selection of a convenient counter to use with a particular quantity is often an important matter. If we combine two quantities whose fraction representation are known and ask for a fraction describing the result, the key to the solution is finding a common denominator, d, for the two fractions. This is needed because $\frac{1}{d}$ is a number describing a quantity which can be used as a counter for both of the original quantities and also for the combined quantity.

The third and final number used to describe a quantity is the numerator of the corresponding fraction, which tells how many of the counter are in the quantity which we are describing. $\frac{3}{4}$ means that in the quantity there are 3 counters, 4 of which would make a unit.

MULTIPLICATION

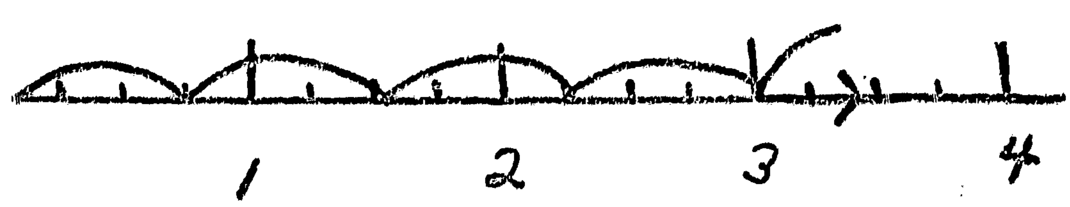
In the realm of whole numbers, we have established a grouping model to help us understand the relation of certain physical world situations to the mathematical operation of multiplication. It is now reasonable to ask how this grouping model can be extended to a pattern which will allow relating similar physical situations involving quantities to multiplication of fractions. Since the number line represents nicely many kinds of quantity, let us use it to represent some fractional groupings.

Making 5 round trips to Grandma's house of $\frac{3}{4}$ miles each gives a total distance of $5 \times \frac{3}{4}$.



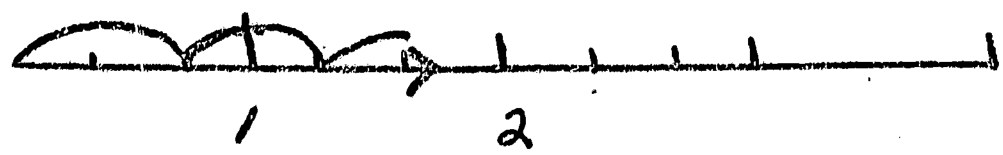
$5 \times \frac{3}{4} = ?$; $? = 3\frac{3}{4}$

If on the fifth trip, we don't come home, the total distance is only $4\frac{1}{2} \times \frac{3}{4}$.

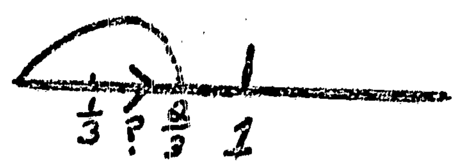


$4\frac{1}{2} \times \frac{3}{4} = ?$; ? = a little more that $3\frac{1}{4}$ miles

Mrs. Quick has a recipe for goulash which calls for $\frac{2}{3}$ cups of minced onion. When she prepares this dish for the whole clan, including grand children, she makes $2\frac{1}{2}$ recipes, which means $2\frac{1}{2} \times \frac{2}{3}$ cups of onion,



$2\frac{1}{2} \times \frac{2}{3} = ?$; It appears that $? = 1\frac{2}{3}$, approximately but when she serves only the three members of the family who still live at home, she makes only $\frac{3}{4}$[1] of a recipe which means $\frac{3}{4} \times \frac{2}{3}$ cups of onion.



$\frac{3}{4} \times \frac{2}{3} = ?$; It appears that $? = \frac{1}{2}$, approximately

In drawing number line pictures for multiplication of fractions such as the ones above, it is convenient to mark the number line with counters indicated by the second factor, for example in showing the multiplication $2\frac{1}{2} \times \frac{2}{3}$ to mark the line into thirds of units. This allows making each hump the right size, and even allows finding an exact answer from the picture when the first factor is a whole number.

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1. The occurrence of the word "of" in a problem is a useful clue that multiplication is involved. But the English language was not invented to facilitate problem solving. "Of", like "take away", is a useful clue but it is not entirely dependable. In the problems above, we want to see a common pattern for $2\frac{1}{2} \times \frac{2}{3} = ?$ and $\frac{3}{4} \times \frac{2}{3} = ?$. The word "of" does not always identify this pattern. It is common to say "$\frac{2}{3}$ of a recipe", but is not common to say "$2\frac{1}{2}$ of a recipe" or even "$2\frac{1}{2}$ of recipes". We say merely "$2\frac{1}{2}$ recipes".

Since the first factor tells how many humps rather than how many units, there is apparently little to be gained by breaking the units on the number line into counters suggested by its denominator. This points up the fact that, when the first factor is not a whole number, number line pictures such as those above cannot be depended upon to produce a precise answer we can multiply using the usual computational method, for example, $\frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{1}{2}$; and $2\frac{1}{2} \times \frac{2}{3} = \frac{5}{2} \times \frac{2}{3} = \frac{10}{6} = \frac{5}{3} = 1\frac{2}{3}$

In the following section we will reconcile this method with quantity model grouping.

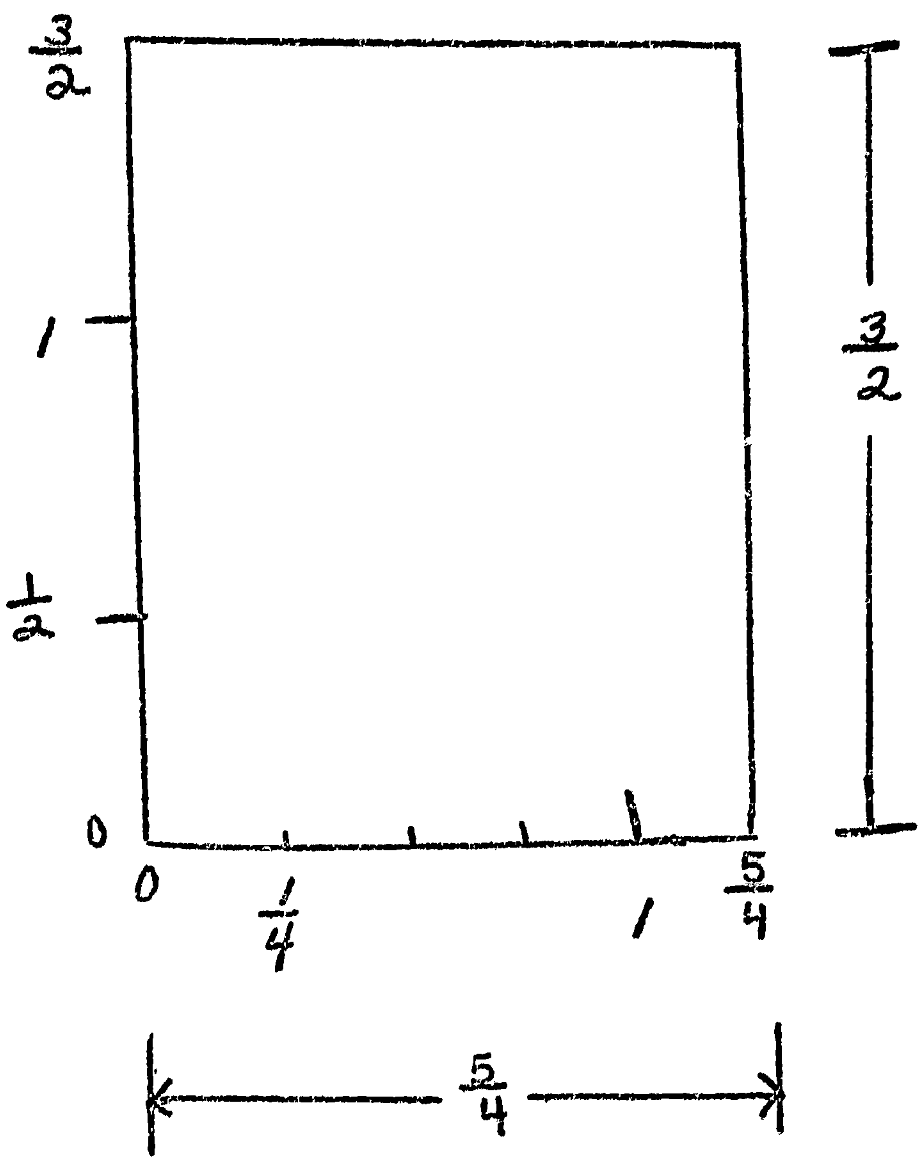
EXERCISES

Draw number line pictures for the following.

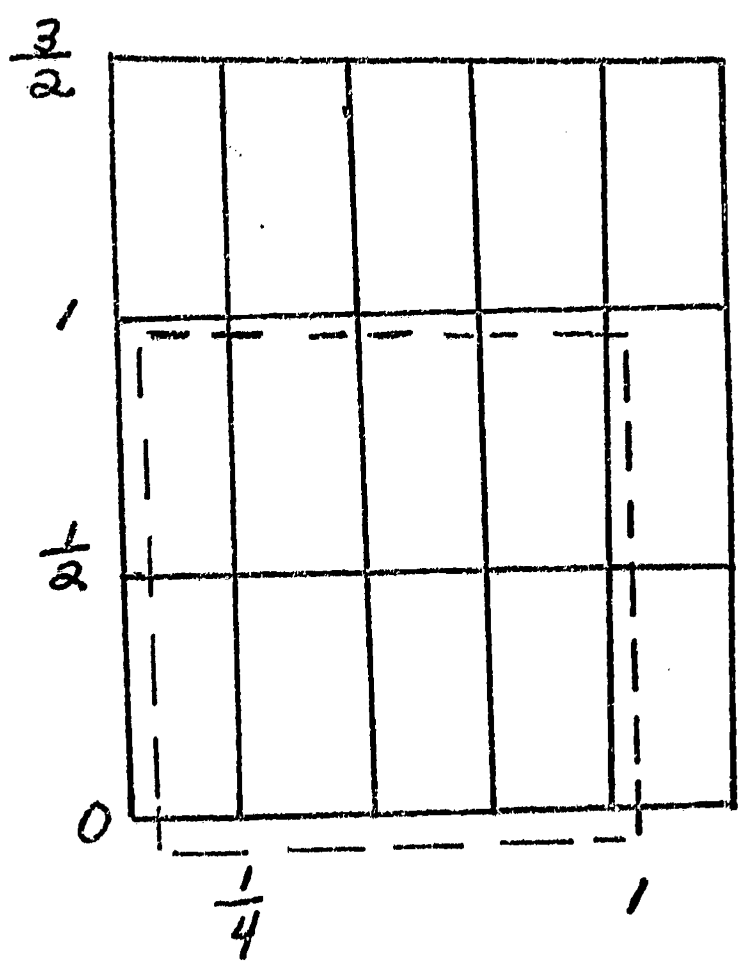
1. $2\frac{1}{2} \times \frac{3}{8} = ?$
2. $\frac{7}{3} \times \frac{1}{4} = ?$
3. $\frac{2}{5} \times 3\frac{2}{3} = ?$
4. $\frac{2}{5} \times \frac{1}{4} = ?$
5. 60% of $24.60 is how much?
6. How far will you get in 3 hours and 20 minutes if you are traveling 50 miles per hour?
7. How much does 2 lb. 4 oz. of steak cost at $1.25 a lb. ?
8. How many pounds in $3\frac{1}{2}$ kilograms? (Use $\frac{11}{5}$ lbs. per kilogram.)
9. Write three problems involving fractional multiplication which can be illustrated using the number line.

USING AREA AND GROUPING MODELS FOR $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

The connection between grouping in the quantity model and the usual "top times top and bottom times bottom" method of computing the product of two fractions can be seen in finding the area of a rectangle. With this end in mind, let us find the area of a $\frac{5}{4}$ by $\frac{3}{2}$ rectangle.

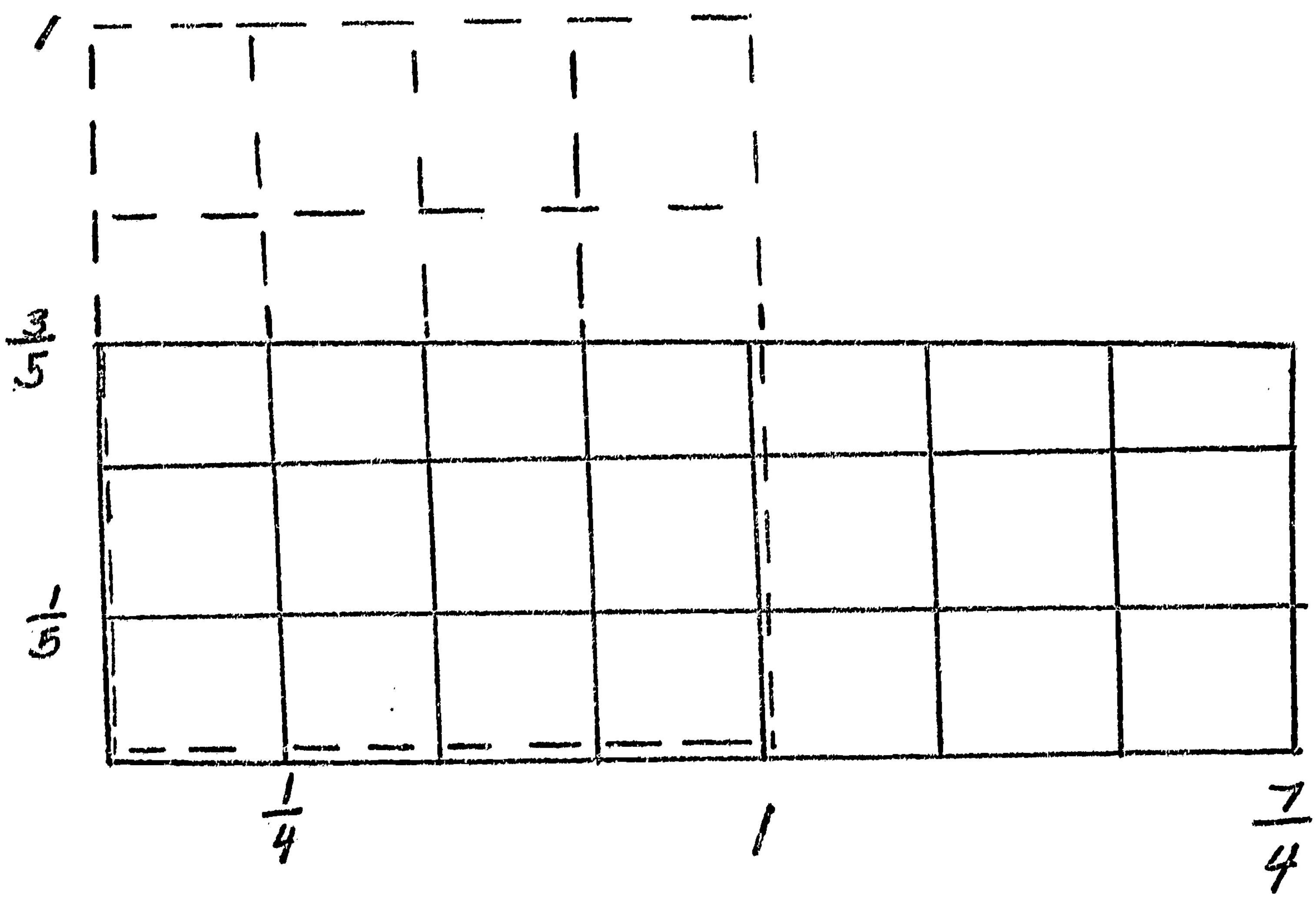


In order to introduce a grouping pattern, we can follow the the suggestion given by $\frac{1}{2}$ and $\frac{1}{4}$, the counters for the two sides, and cut the rectangle into smaller equal rectangles, $\frac{1}{2}$ by $\frac{1}{4}$.



To use the small rectangles as counters, we note that there are 4 columns of 2 each of them in a unit of area (marked with dashes), so that each counter is $\frac{1}{4X2}$ units (square inches) of area. In the whole rectangle, there are 5 columns of 3 each of these counters, so that $\frac{5X3}{4X2}$ describes the total area.

In multiplication such as $\frac{3}{5} \times \frac{7}{4}$, the same analysis is possible, except that a square picture of a unit of area cannot be drawn inside a $\frac{3}{5}$ by $\frac{7}{4}$ rectangle. We can however extend the picture to include a square unit so that we can find the number of area counters in it.



In this picture, there are 5×4 area counters in a unit of area. There are 3×7 area counters in the $\frac{3}{5}$ by $\frac{7}{4}$ rectangle. Thus the area of the rectangle is $\frac{3 \times 7}{5 \times 4} = \frac{21}{20}$ sq. units.

EXERCISES

1. Draw the diagrams you would use in explaining with rectangles why each of the following is true.

(a) $\frac{3}{2} \times \frac{7}{5} = \frac{3 \times 7}{2 \times 5}$ (b) $\frac{3}{2} \times \frac{4}{5} = \frac{3 \times 4}{2 \times 5}$

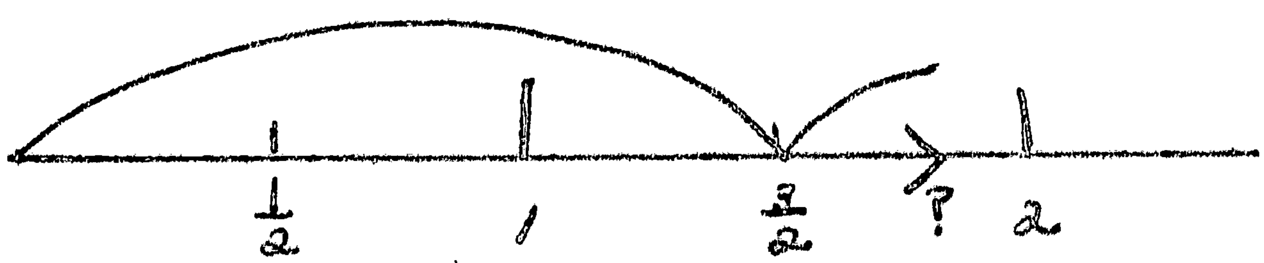
(c) $\frac{1}{3} \times \frac{1}{4} = \frac{1 \times 1}{3 \times 4}$

2. Draw a rectangle picture for $\frac{3}{2} \times \frac{5}{6}$. To explain why the "cancellation" $\frac{\cancel{3}^{1}}{2} \times \frac{5}{\cancel{6}_{2}}$ works, show how counters of size $\frac{1}{2 \times 2}$ can be used to find the area. Do the same for $\frac{9}{2} \times \frac{4}{3}$. (The counters may not be rectangular.)

3. Using a rectangular picture of $\frac{5}{4} \times \frac{3}{2}$, explain why the commutative law is true for multiplication of fractions i.e., show that $\frac{5}{4} \times \frac{3}{2} = \frac{3}{2} \times \frac{5}{4}$.

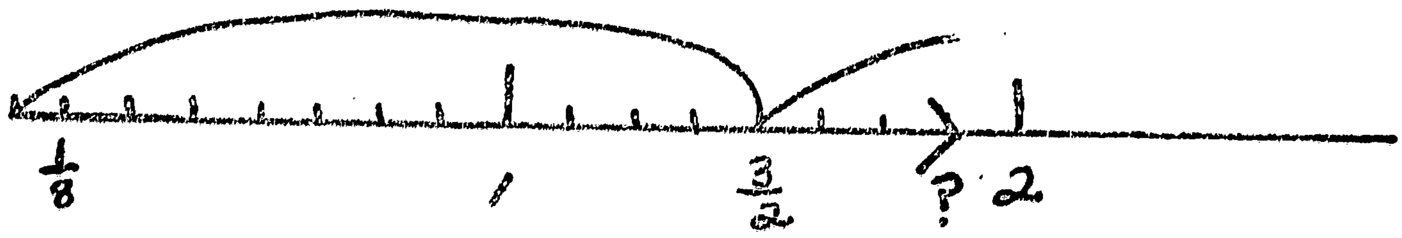
MODELS FOR $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ USING THE NUMBER LINE

The demonstration in the preceeding section that our method of multiplying fractions does what we expect to do in grouping situations depends rather heavily on the rectangular diagrams. The unmentioned shift from a unit of length to a unit of area makes the explanation rather difficult to extend to non-area grouping. In order to see how the argument can be made within the framework of a more generally applicable model, let us investigate why $\frac{5}{4} \times \frac{3}{2} = \frac{5 \times 3}{4 \times 2}$, using the number line.

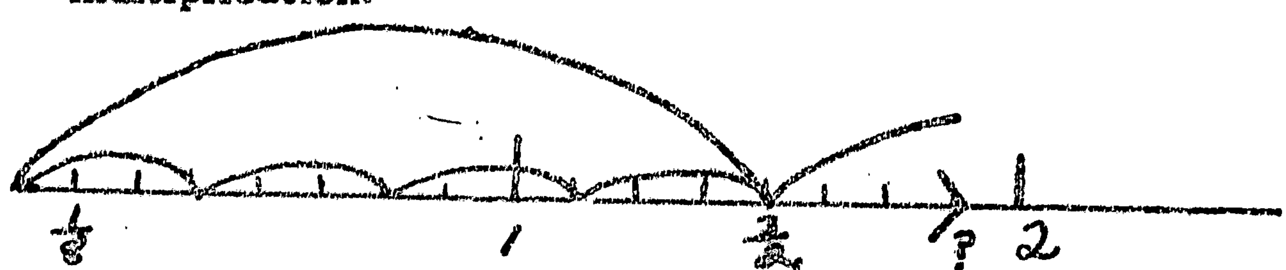


$\frac{5}{4} \times \frac{3}{2} = ?$

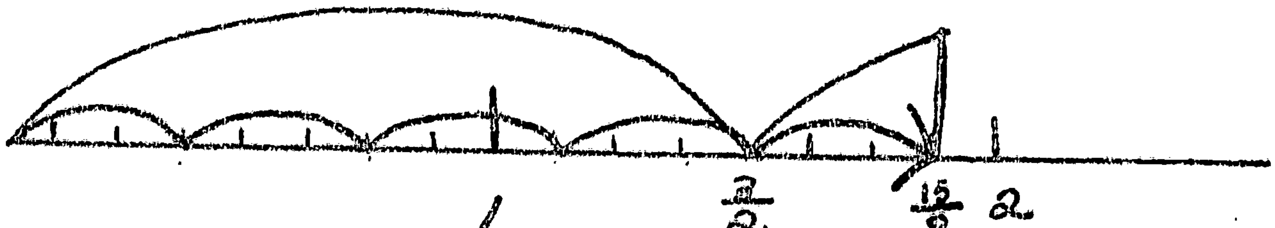
As suggested earlier, the main difficulty in seeing how a precise answer can be obtained by looking at this picture is that $\frac{5}{4}$ describes "humps", not units. If we knew how many <u>units</u> $\frac{1}{4}$ of a <u>hump</u> was, we could take 5 of this amount, and have an answer. Using our experience with area to suggest counters of $\frac{1}{4X2} = \frac{1}{8}$ units for the size of a counter, let us attack this difficulty.



We see that there are 12 of these counters in one hump. In terms of counters, then, $\frac{1}{4}$ of a hump is a segment of length "?" such that $4 \times ? = 12$ counters. This is a whole number partition division of a set of counters, a picture of which we can superimpose on our original multiplication.



Putting things back together, five $\frac{1}{4}$'s of a hump is 5×3 counters or 15 counters.



Since there are 4×2 counters in a unit, $\frac{5}{4} \times \frac{3}{2} = \frac{5X3}{4X2}$.

Of historical interest is the following explanation of multiplication of fractions:

"Suppose we are required to multiply 4/5 by 2/3.

In this figure let the line AB be divided into five equal parts at the points C, D, E, and F. Then AF is 4/5 of AB.

That is, 1/3 of 1/5 = 1/15 of the whole.
Then 1/3 of 4/5 must be 4 times as much, or 4/15.
Then 2/3 of 4/5 must be twice 4/15, or 8/15.

Therefore, to multiply a fraction by a fraction, find the product of the numerators for the required numerator and the product of the denominators for the required denominator.

EXERCISES

Use the number line to show that

1. $\frac{7}{5} \times \frac{2}{3} = \frac{14}{15}$ 2. $\frac{6}{5} \times \frac{4}{3} = \frac{24}{15}$.

* George Wentworth and David Eugene Smith, Complete Arithmetic, Part I, Boston: Ginn and Company, 1909, pp. 102.

CONCEPTUAL MODELS

IN

TEACHING THE USES OF NUMBER AND OPERATION

USOE Project

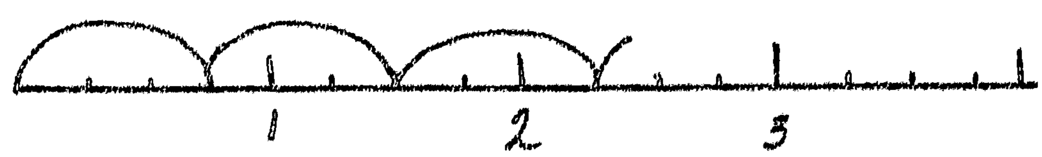
R. G. Clason

Unit 4. A Quantity Model for Division of Fractions

MEASUREMENT DIVISION

As in the case of whole number division without a remainder, division of fractions in the quantity model follows the measurement and partition patterns. In the measurement, we start with a fraction for the size of each group and a fraction for the total quantity to be divided, and look for a fraction describing the number of groups.

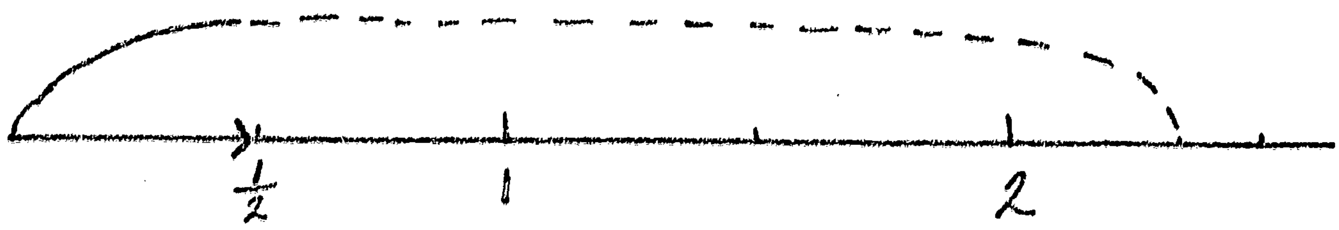
Consider the measurement division $? \times \frac{3}{4} = 3\frac{2}{3}$. As in the whole number case, we can use the known size of a group to construct a picture by making one hump at a time.



$? \times \frac{3}{4} = 2\frac{2}{3}$; $?$ = a little more than 3 humps.

In order to keep things simple and to stress the grouping pattern, this picture is sketched without doing any computation. Because of this, a choice must be made between counters of length $\frac{1}{4}$ units and $\frac{1}{3}$ units to be shown on the number line. $\frac{1}{4}$, the counter needed to construct a group, is used because the group size is used repeatedly, while the size of the total appears only once in the diagram.

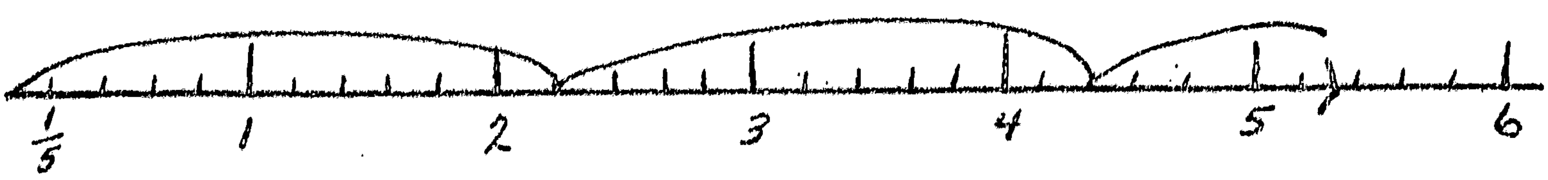
The number line picture of a measurement division has a somewhat different appearance when the group is larger than the total. For $? \times \frac{7}{3} = \frac{1}{2}$, the picture can be drawn



$? \times \frac{7}{3} = \frac{1}{2}$; $\quad ? =$ about $\frac{1}{5}$ of a group

so that, even though the total does not include a whole group, a whole group is shown.

As an example of the solution of a problem using the measurement division pattern, let us change $5\frac{1}{3}$ lbs. to kilograms. There are about $\frac{11}{5}$ lbs. in one kilogram, so using lbs. for units, we must find how many groups of $\frac{11}{5}$ lbs. each are in $5\frac{1}{3}$ lbs.



$? \times \frac{11}{5} = 5\frac{1}{3}$; $\quad ? =$ almost $3\frac{1}{2}$ groups (kilograms)

EXERCISES

Draw number line pictures for the following.

1. $? \times \frac{2}{3} = 3\frac{2}{5}$

2. $? \times \frac{7}{2} = 11\frac{1}{2}$

2. $? \times \frac{7}{3} = \frac{2}{5}$

4. $? \times \frac{2}{3} = \frac{7}{3}$

5. How many kilograms in $\frac{1}{2}$ lb ?

6. How many lbs. in $\frac{1}{2}$ kilogram ?

7. $\frac{5}{2}$ cm. is about equal to one in. How many cm. in $5\frac{1}{3}$ inches?

8. How many inches in $5\frac{1}{3}$ cm?

9. Work problems 5 and 6 using one lb. equal about $\frac{5}{11}$ kilograms, and work problems 7 and 8 using one cm. equals about $\frac{2}{5}$ in.

10. Mrs. Quick has forgotten how many goulash recipes to make for the whole clan, but she remembers that she uses $1\frac{2}{3}$ cups of minced onion. Her recipe book calls for $\frac{2}{3}$ cups of onion. How many recipes should she make?

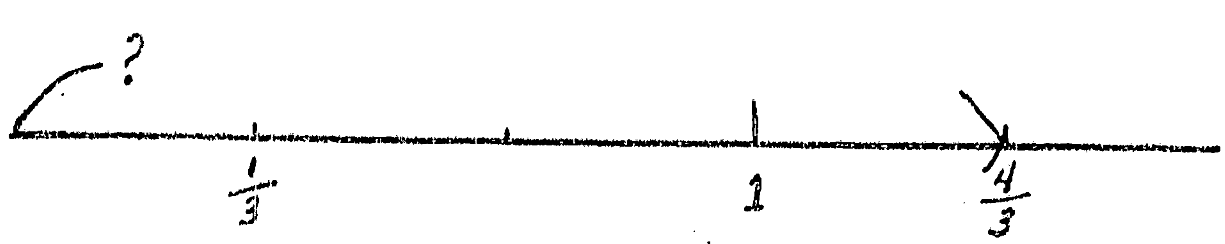
11. Mr. Checkit gets 18.5 miles per gallon from his car on trips. How many gallons of gas will be need to make a trip of 200 miles?

12. Write three problems that can be solved using fractions and measurement division.

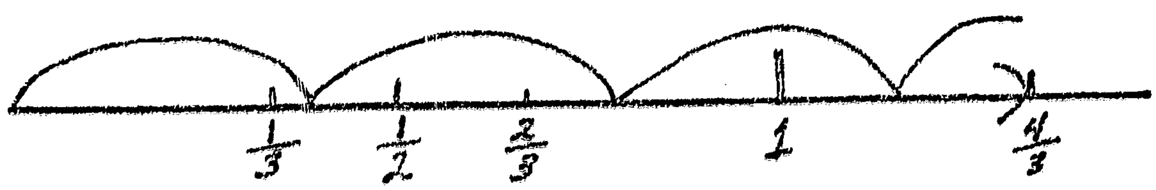
13. How is the number line picture for problem # 4 above similar to the picture for ? x 2 = 7 ?.

PARTITION DIVISION

As with partition division of whole numbers, sketching a number line picture for a partition division for fractions representing quantities is complicated because the size in units of one hump is the unknown. To show $3\frac{1}{2} \times ? = \frac{4}{3}$ we must construct $3\frac{1}{2}$ humps above a line segment $\frac{4}{3}$ units long.



One cannot be very accurate in constructing these humps without some computation, but in reasonably simple cases, a sketch can be made well enough to illustrate the grouping pattern and to get an approximate fraction for the number of units in each group.

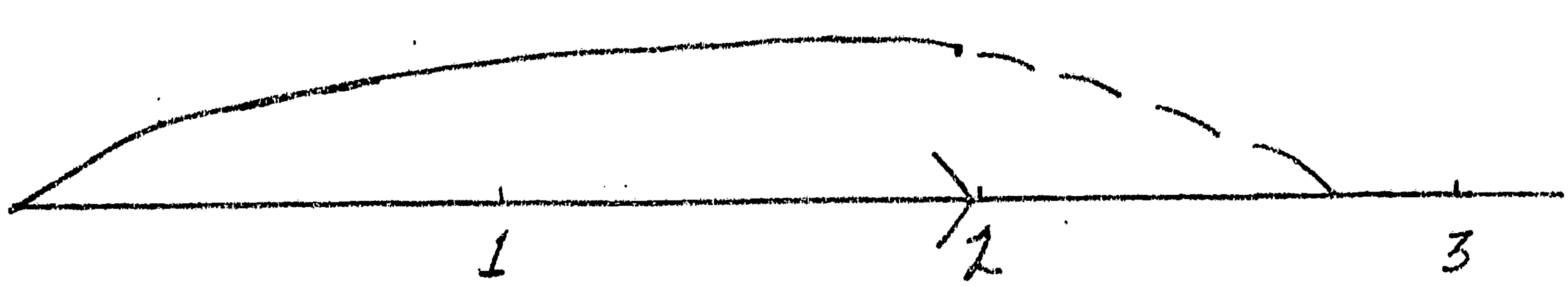


$3\frac{1}{2} \times ? = \frac{4}{3}$, $\quad ? =$ a little less than $\frac{1}{2}$ units in a group

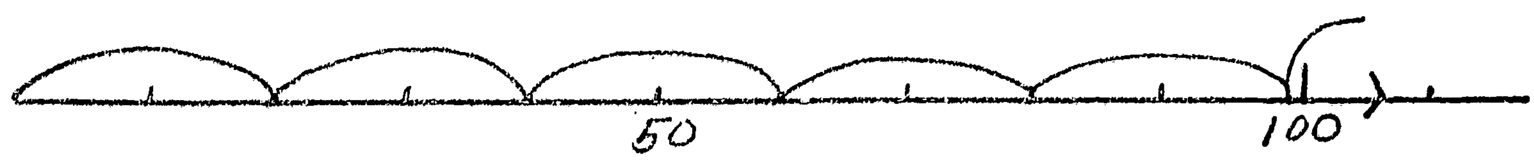
Note that the counters used in this picture are $\frac{1}{3}$ units, the counter for the total. This is the only counter available which measures units; the $\frac{1}{2}$'s in the $3\frac{1}{2}$ describe groups, not units.

Since the question in partition division is "how many units in one group?", if the total does not include at least one full hump on the number line picture, it is desirable to include a whole hump in the picture anyway. For example, in the division $\frac{3}{4} \times ? = 2$,

we want to sketch a hump so that $\frac{3}{4}$ of it is 2 units. Thus one hump must be bigger than 2 units.



Gas mileage problems can be solved using the partition division model. Mr. Checkit finds that his car went 105 miles on 5.5 gallons of gas. On the average, how far did his car go on each gallon?



5.5 x ? = 105 ; ? = about 20 units (miles) for each group (gallon). In this diagram, each hump represents a gallon's worth of distance.

EXERCISES

Draw a number line picture for each of the following. Re-write 1-4 as division statements.

1. $1\frac{1}{2} \times ? = \frac{3}{4}$ 2. $? \times \frac{3}{4} = 1\frac{1}{2}$

3. $\frac{3}{4} \times ? = 1\frac{1}{2}$ 4. $? \times 1\frac{1}{2} = \frac{3}{4}$

5. $1\frac{1}{2} \times \frac{3}{4} = ?$ 6. $\frac{3}{4} \times 1\frac{1}{2} = ?$

7. What is the gas mileage for a car that goes 175 miles on 5.5 gallons of gas?

8. Mrs. Quick has lost her cookbook. She wants to make one recipe of goulash. In searching for the book, she comes across a slip of paper on which she has jotted down the various amounts of ingredients needed for $2\frac{1}{2}$ recipes. If she needed $1\frac{2}{3}$ cups of minced onion for $2\frac{1}{2}$ batches, how much onion will she need for one recipe?

9. $\frac{2}{5}$ oz. of Essense of 1002 Lilacs costs \$ 4.50. At this price how much would one oz. cost?

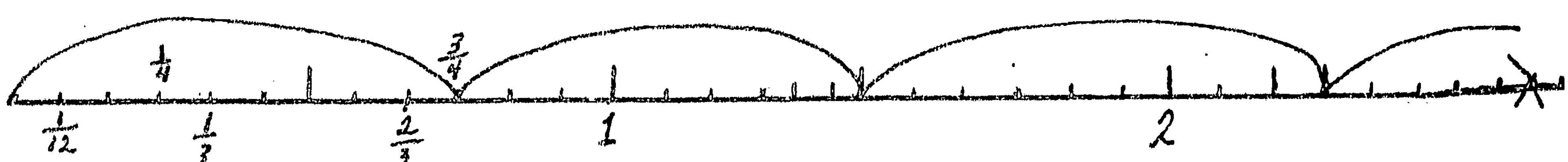
10. Write three problems that can be solved using the partition model for division of fractions.

COMMON DENOMINATOR METHOD OF COMPUTATION

As with multiplication of fractions, it is possible to reconcile computational methods for dividing fractions with the quantity-grouping model. We will consider two ways of doing this, the common denominator method stemming from measurement division, and the invert and multiply method stemming from partition division.

When we first looked at measurement division of fractions on the number line using the example $? \times \frac{3}{4} = 2\frac{2}{3}$, we failed to get an exact answer from the number line picture because we did not want to complicate the basic grouping pattern by introducing counters of both $\frac{1}{3}$ and $\frac{1}{4}$ units. Let us now accept this complication and proceed by choosing a new counter which will count segments of both thirds and fourths of units. A counter that will work is one of length $\frac{1}{4 \times 3}$ or $\frac{1}{12}$.

$? \times \frac{3}{4} = 2\frac{2}{3}$ becomes $? \times \frac{9}{12} = 2\frac{8}{12}$ or $? \times \frac{9}{12} = \frac{32}{12}$.



$? \times \frac{9}{12} = \frac{32}{12}$; $? \times 9 = 32$; $? = \frac{32}{9}$ humps.

We can see from the picture that there are 3 humps and 5 counters ($\frac{1}{12}$ units each) more. But we are answering the question "how many groups?" Since there are 9 counters in a group, the 5 counters represent $\frac{5}{9}$ of a group, and the answer is $3\frac{5}{9}$ groups.

In this demonstration, we first describe the total length and the size of one group oin two ways: in units and in counters of $\frac{1}{12}$ units. With either description, we see that the number of humps is the same. As a result, we can phrase the question in terms of counters rather than units, thus replacing the fractional division $? \times \frac{9}{12} = \frac{32}{12}$ with the whole number division $? \times 9 = 32$. This leads to the common denominator method of dividing fractions:

$$2\frac{2}{3} \div \frac{3}{4} =$$

$$\frac{8}{3} \div \frac{3}{4} =$$

$$\frac{32}{12} \div \frac{9}{12} =$$

$$32 \div 9 = \frac{9}{32}$$

EXERCISES

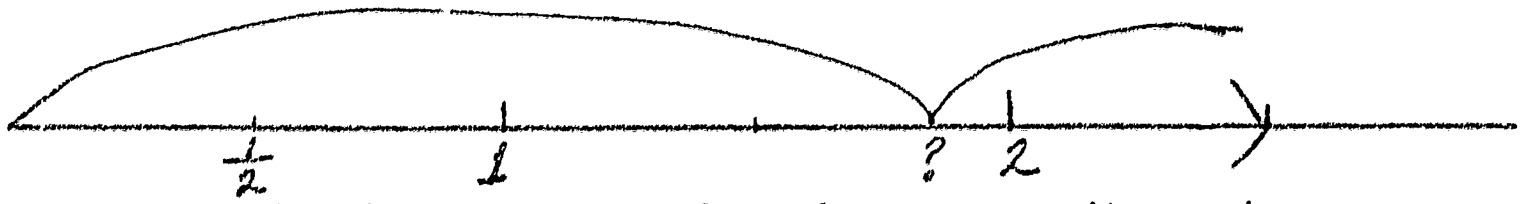
Using the following problems rewrite each as a division problem, then draw a number line picture to illustrate why the common denominator method of division works.

1. $? \times \frac{2}{3} = \frac{4}{5}$

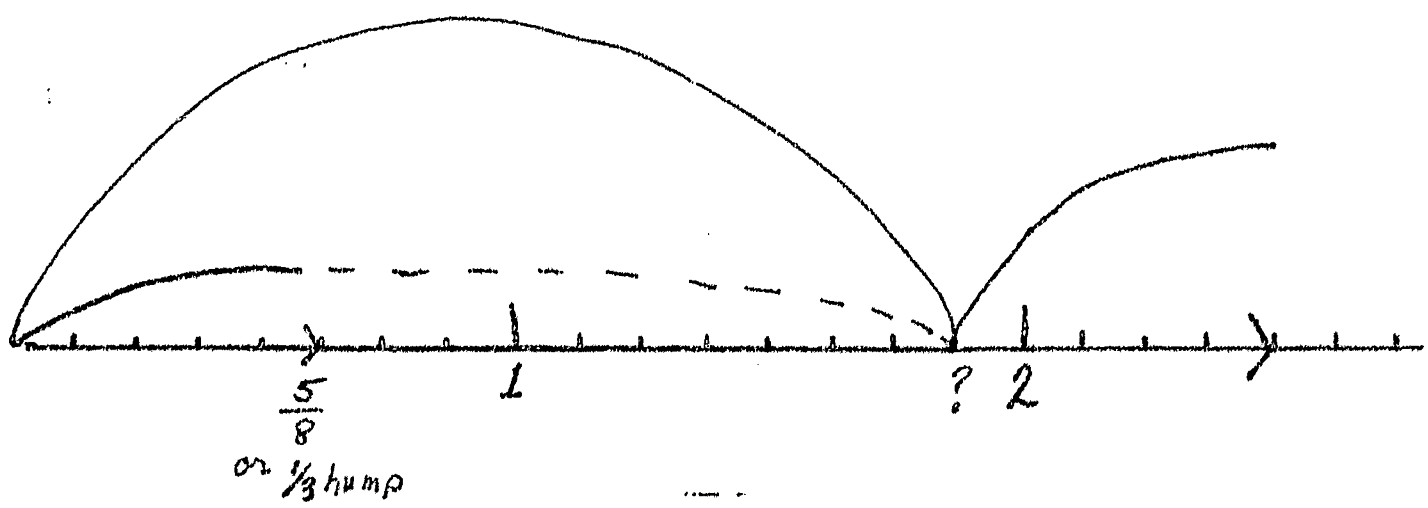
2. $? \times 2\frac{1}{2} = 4$.

INVERT AND MULTIPLY METHOD OF COMPUTATION

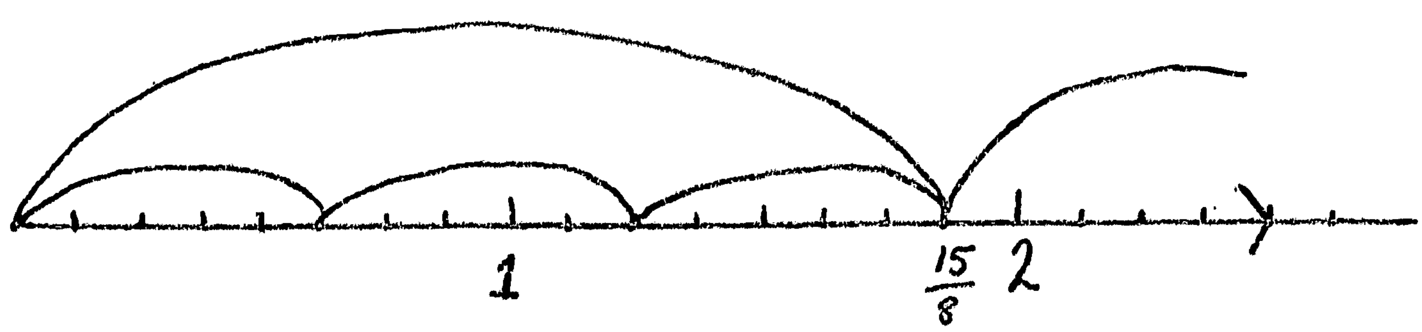
Using partition division, we can produce a different method for dividing fractions. As an example, let us solve $\frac{4}{3} \times ? = \frac{5}{2}$.



In this division, we want to know how many units are in one group. We know to begin with that $\frac{4}{3}$ humps represents the same quantity as $\frac{5}{2}$ units. If we find $\frac{1}{4}$ of the total quantity with respect to both descriptions, we find that $\frac{1}{4}$ of $\frac{4}{3}$ humps represents the same quantity as $\frac{1}{4}$ of $\frac{5}{2}$ units, hence, $\frac{1}{3}$ hump represents the same quantity as $\frac{5}{4 \times 2}$ units. We can superimpose a picture of this multiplication on our division picture:



Since we are interested in the size of one hump, let us multiply both descriptions of $\frac{1}{3}$ hump ($\frac{5}{8}$ unit) by 3: $3 \times \frac{1}{3}$ hump represents the same quantity as $3 \times \frac{5}{4 \times 2}$ units.



Hence, one hump represents the same quantity as $\frac{3 \times 5}{4 \times 2}$. Thus one hump contains $\frac{3 \times 5}{4 \times 2}$ units, so that starting with the division $\frac{5}{2} \div \frac{4}{3} = ?$ we find that $? = \frac{3 \times 5}{4 \times 2}$ or $? = \frac{5}{2} \times \frac{3}{4}$. Thus our original problem has proceeded from $\frac{4}{3} \times ? = \frac{5}{2}$ to $? = \frac{5}{2} \div \frac{4}{3}$ to $? = \frac{5}{2} \times \frac{3}{4}$. This is a particular example of the old algorithm: To divide one rational number by a second invert the divisor (that is take its reciprocal) and multiply.

EXERCISES

Using the following divisions, draw number line pictures to illustrate that the invert and multiply method of dividing fractions works.

1. $\frac{3}{2} \times ? = \frac{5}{3}$. 2. $\frac{3}{2} \times ? = \frac{3}{5}$.

CONCLUSION

All applications of mathematics involve the idea of associating elements of our non-mathematical physical or mental environment with the elements of mathematical systems. Operations and problem solving processes are then carried out within the mathematical system and re-interpreted back into the context of the original problem. The more this stratigy is understood the more people will understand both the nature of mathematics and the processes by which it is used.

Somewhat in reverse, the building of an understanding of a mathetical system, in our case of the operations of multiplication and division, may be greatly assisted by working with concrete and conceptual models of the logical abstraction which is the mathematics. These units have reviewed different models and ways in which they may be used to discover or clarify the algorithms for and meaning of the operations of multiplication and division.

The possession of an understanding of some basic or primary model as well as of a rule helps students to recall and to apply properly the rule. Such a basic conceptual model is an important tool for the teacher who can direct an individual student or a class back to it either to clarify a new problem or as a beginning for a series of thought-steps and problems which will lead to a new extension or process.

This basic reference concepts should in time become familiar generalizations such as the distributive law, the structure of a field, or the notion of an inverse operation. However, they may well begin as more concrete conceptual models such as the "take-away" or "part-part-whole" views of subtraction and the idea of multiplication as "number of groups times the number in a group is the total number" as discussed above. Such early concrete conceptual models serve as a basis for applying mathematics, for solving problems, and for extending our mathematical system toward its ultimately more abstract and general structures.

Whenever the development and learning process comes to this stage, however, the cycle should be completed by returning to concrete or special situations in which newly developed generalizations or algorithms can be perceived and used in another physical or conceptual environment. Ultimately students should not have to talk or think of "humps" to complete mathematical problems or applications. Such models used early may help many students to move toward an ultimately deeper understanding.